Journal of Sound and Vibration (2001) **246**(1), 157–174
doi:10.1006/jsvi.2001.3625, available online at http://www.idealibrary.com on IDEAL®

DYNAMIC ANALYSIS OF CONTINUOUS PLANE TRUSSES WITH EQUIDISTANT SUPPORTS

C. W. CAI AND J. K. LIU

Department of Mechanics, Zhongshan University, Guangzhou 510275, *People's Republic of China*

AND

F. T. K. AU AND L. G. THAM

Department of Civil Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong.
E-mail: *hrecatk@hku.hk*

(*Received* 16 *March* 2000, *and in final form* 23 *February* 2001)

The natural and forced vibrations of a continuous Warren truss resting on equidistant roller supports are studied. In the analyses, the longitudinal inertial forces are neglected in the equation of motion and the mass of the truss is replaced by two sets of lumped masses at the nodes. In the forced vibration case, the truss is subjected to transverse harmonic loads acting at the nodes. The truss can be considered as an equivalent system with cyclic bi-periodicity: periodicities due to the repetitive units of the truss and the supports respectively. Therefore, the governing equation for such a system can be uncoupled by applying U-transformation twice to form a set of single-degree-of-freedom equations. Such equations can be used to solve for the natural frequencies of the trusses or their responses due to external excitations. Typical Warren trusses are taken as examples to demonstrate the advantages of the proposed method and the accuracy that can be achieved. The response of a truss with six substructures and four supports with a harmonic load acting at the centre node is investigated.

1. INTRODUCTION

Bi-periodic structures were investigated by many researchers using various methods, including transfer matrix method [1], wave approach [2–4], standard stiffness and transmission methods [5], and U-transformation method [6, 7]. The bi-periodic Warren truss considered in the present study is a periodic structure with periodic supports. The period of the truss is normally different from that of the supports.

The transverse vibration of a Warren truss with two simply supported ends and lumped masses at the nodes was investigated by Cai *et al.* [8] using U-transformation technique [9–11] where the truss was regarded as a single periodic structure. More recently, the static analysis of a Warren truss with equidistant roller supports subjected to transverse loading has been solved by Cai *et al.* [12] using the same technique. Details of U-transformation and its applications are given in the book by Chan *et al.* [13]. In this paper, the U-transformation technique is applied to the natural and forced vibration analyses of a continuous Warren truss with periodic roller supports. As an example, a Warren truss with $2n$ substructures and $n+1$ supports is considered. The numerical results of natural frequencies for $n = 2, 3, 4, 5, 6, 30, \infty$ are found from the frequency equations. The response

0022-460X/01/360157 + 18 $35.00/0

of a truss with six substructures and four supports under the action of loading applied at the centre node is also studied.

The proposed dynamic analysis approach has many obvious advantages over the numerical methods in the optimal design, sensitivity analysis and computing cost. It must be pointed out that it is not easy to obtain accurate results using numerical methods in some cases because the natural frequencies of a periodic structure are densely populated. Therefore, the frequencies could be fairly close. In the present approach, the frequency equation of a continuous truss with n spans is uncoupled to form n individual frequency equations and the natural frequencies obtained from each equation are dispersed. Consequently, accurate natural frequencies can be easily found even if n is large.

2. FORMULATION OF THE PROBLEM

Consider a Warren truss with equidistant roller supports subjected to transverse nodal loads (Figure 1). The truss consists of a linear assembly of identical four-bar sets, as shown in Figure 2(b), pin-joined at the nodes. Only axial forces act on these bars. The bars in the longitudinal direction are assumed to have modulus of elasticity E_1, cross-sectional area A_1 and length L_1 while the inclined bars have modulus of elasticity E_2, cross-sectional area A_2 and length L_2. Furthermore, the masses of the bars are assumed to be lumped at the nodes. They are denoted by M_1 and M_2, and attached to each of the lower and upper nodes respectively (Figure 1). In Figure 1, the two integers N and $n + 1$ denote the total numbers of substructure units and supports respectively. The integer p denotes the number of substructure units between two adjacent supports.

The Warren truss is composed of a number of repetitive substructure units and a typical substructure unit is shown in Figure 2(a). Each substructure unit consists of four bars and four nodes. The serial number of the node is made up of two integers in which the first one denotes the ordinal number of the node in the substructure unit and the second one is the ordinal number of the substructure unit. The serial numbers of the nodes are given in round brackets. In order to avoid repetition in the analysis, we consider only the masses and the nodal loads of two nodes on the left of every substructure unit.

Strictly speaking, such a truss is not a cyclic bi-periodic structure. However, we can convert it into an equivalent cyclic bi-periodic structure so that the U-transformation approach can be applied to uncouple the equation of motion. First, the truss is extended by its symmetrical image and an anti-symmetric loading is applied on the corresponding extended part as shown in Figure 3. The two bars denoted by dotted lines (Figure 3) are additional ones without masses. If the two pairs of extreme nodes a–a' and b–b' are imaginarily joined by hinges, the first substructure unit is connected to the last one and such an extended system possesses cyclic periodicity. Under conditions of antisymmetric vibration, it can be readily shown that the two additional bars are not subjected to any load.

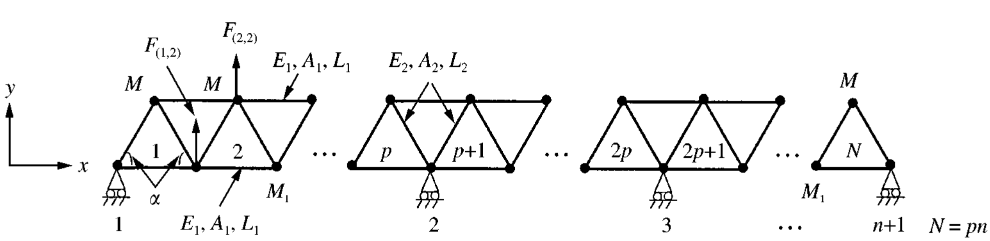

Figure. 1. Plane truss with equidistant roller supports.

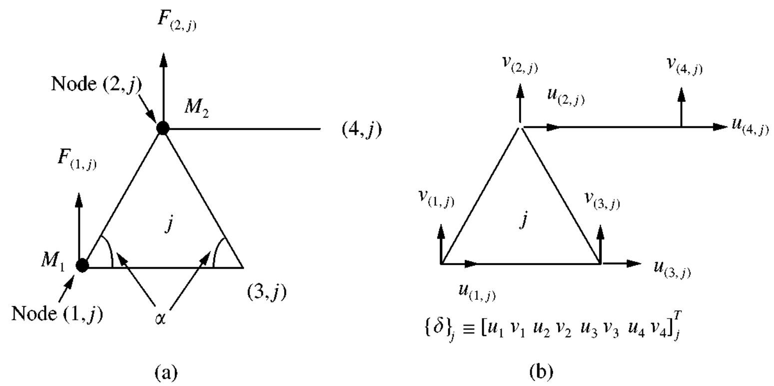

Figure. 2. Substructure unit. (a) The serial numbers of the nodes. (b) The displacements of substructure.

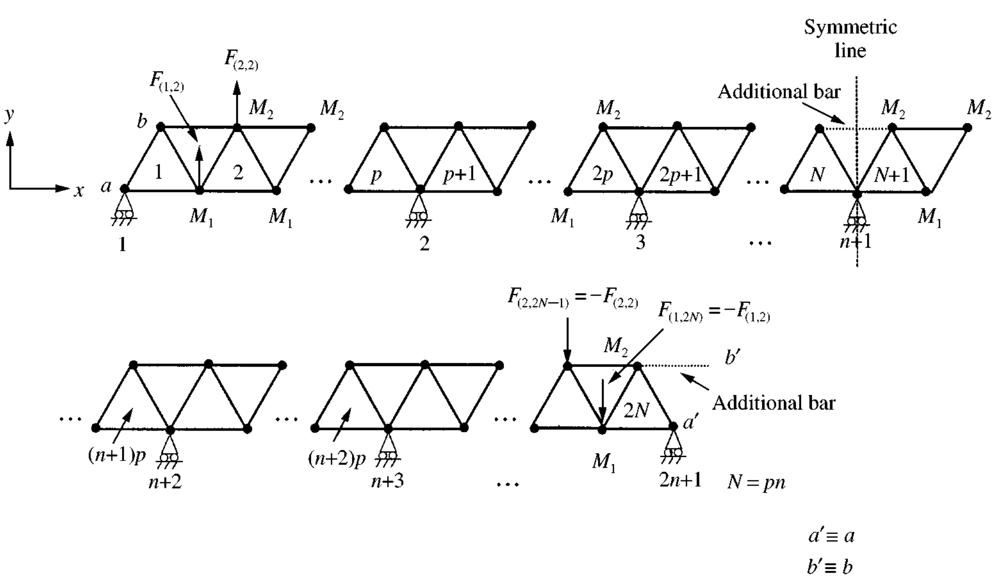

Figure. 3. Equivalent system with cyclic bi-periodicity subjected to antisymmetric loads.

As a result, the antisymmetric vibration of the extended truss will not be affected by such additional bars and the extended truss is equivalent to the original one.

Secondly, the supports are replaced by support reactions which can be determined by the constraint conditions (zero displacements) at supports. Subsequently, the equivalent truss can be treated as a cyclic single periodic structure. Such a structure can be analyzed by means of the U-transformation technique [8].

To carry out analyses of such a structure, one has to define the total potential energy and kinetic energy of the system. They can be defined as follows.

2.1. TOTAL POTENTIAL ENERGY

The total potential energy V of the equivalent system with $2N$ substructure units can be expressed as

$$V = \sum_{j=1}^{2N} V_j, \tag{1}$$

where V_j denotes the total potential energy of the jth substructure unit. In general, V_j is

$$V_j = \frac{1}{2}\{\bar{\delta}\}_j^{\mathrm{T}}[K]_{sub}\{\delta\}_j - \frac{1}{2}(\{\bar{\delta}\}_j^{\mathrm{T}}\{F\}_j + \{\delta\}_j^{\mathrm{T}}\{\bar{F}\}_j), \tag{2}$$

where $[K]_{sub}$ denotes the stiffness matrix of the substructure, $\{\delta\}_j$ and $\{F\}_j$ denote the displacement and load vectors for the jth substructure unit, respectively, and the superior bar indicates complex conjugation.

For the present case, $\{\delta\}_j$ can be defined as

$$\{\delta\}_j = \begin{Bmatrix} \delta_L \\ \delta_R \end{Bmatrix}_j, \tag{3a}$$

where

$$\{\delta_L\}_j = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}_j, \qquad \{\delta_R\}_j = \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_j \tag{3b}$$

with its components shown in Figure 2(b). The parameters u and v denote the longitudinal and transverse displacement components respectively.

As two adjacent substructure units must be continuous, we have

$$\{\delta_R\}_j = \{\delta_L\}_{j+1}, \quad j = 1, 2, \ldots, 2N. \tag{4a}$$

Due to cyclic periodicity, one can show that

$$\{\delta_L\}_{2N+1} \equiv \{\delta_L\}_1. \tag{4b}$$

For each substructure unit, the stiffness can be expressed in matrix form as

$$[K]_{sub} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \tag{5}$$

where

$$[K_{11}] = \begin{bmatrix} K_1 + K_2\cos^2\alpha & K_2\sin\alpha\cos\alpha & -K_2\cos^2\alpha & -K_2\sin\alpha\cos\alpha \\ K_2\sin\alpha\cos\alpha & K_2\sin^2\alpha & -K_2\sin\alpha\cos\alpha & -K_2\sin^2\alpha \\ -K_2\cos^2\alpha & -K_2\sin\alpha\cos\alpha & K_1 + 2K_2\cos^2\alpha & 0 \\ -K_2\sin\alpha\cos\alpha & -K_2\sin^2\alpha & 0 & 2K_2\sin^2\alpha \end{bmatrix},$$

$$[K_{12}] = [K_{21}]^{\mathrm{T}} = \begin{bmatrix} -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -K_2\cos^2\alpha & K_2\sin\alpha\cos\alpha & -K_1 & 0 \\ K_2\sin\alpha\cos\alpha & -K_2\sin^2\alpha & 0 & 0 \end{bmatrix},$$

$$[K_{22}] = \begin{bmatrix} K_1 + K_2\cos^2\alpha & -K_2\sin\alpha\cos\alpha & 0 & 0 \\ -K_2\sin\alpha\cos\alpha & K_2\sin^2\alpha & 0 & 0 \\ 0 & 0 & K_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in which $K_1 \equiv E_1A_1/L_1$, $K_2 \equiv E_2A_2/L_2$ and α denotes the angle of inclination of the inclined bars as shown in Figure 1.

As external excitations as well as support reactions are acting on the truss, the load vector can be written as

(1) *Unsupported nodes*

$$\{F\}_j = [0\ F_{(1,j)}\ 0\ F_{(2,j)}\ 0\ 0\ 0\ 0]^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega t}, \quad j \neq 1 + (k-1)p,\ k = 1, 2, \ldots, 2n. \tag{6a}$$

(2) *Supported nodes*

$$\{F\}_j = [0\ F_{(1,j)} + P_k\ 0\ F_{(2,j)}\ 0\ 0\ 0\ 0]^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega t}, \quad j = 1 + (k-1)p,\ k = 1, 2, \ldots, 2n, \tag{6b}$$

where P_k indicates the amplitude of the support reaction at the kth support; $F_{(1,j)}$ and $F_{(2,j)}$ denote the amplitudes of the external excitations acting at the nodes $(1, j)$ and $(2, j)$, respectively; and ω and t denote the frequency and time variable respectively. In the above expressions, the longitudinal forces are assumed to be zero.

As the external excitations acting on the extended truss are antisymmetric by definition, one can show that

$$F_{(1,2N-j+2)} = -F_{(1,j)}, \quad j = 2, 3, \ldots, N, \tag{7a}$$

$$F_{(2,2N-j+1)} = -F_{(2,j)}, \quad j = 1, 2, \ldots, N, \tag{7b}$$

$$F_{(1,1)} = F_{(1,N+1)} = 0 \tag{7c}$$

in which $F_{(1,j)}$ $(j = 2, 3, \ldots, N)$ and $F_{(2,j)}$ $(j = 1, 2, \ldots, N)$ are the amplitudes of the external excitations acting on the original truss.

2.2. KINETIC ENERGY

The kinetic energy T of the equivalent truss can be expressed as

$$T = \sum_{j=1}^{2N} T_j, \tag{8}$$

where T_j denotes the kinetic energy of the jth substructure unit. One can show readily that

$$T_j = \tfrac{1}{2}\{\dot{\delta}\}_j^{\mathrm{T}} [M]_{sub} \{\dot{\delta}\}_j, \tag{9}$$

where $[M]_{sub}$ denotes the mass matrix of a single substructure unit and the dot denotes differentiation with respect to the time variable t.

If the inertial forces in the longitudinal direction are neglected, $[M]_{sub}$ can be expressed as

$$[M]_{sub} = \begin{bmatrix} \mathrm{M}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{10}$$

where

$$[M_{11}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 \end{bmatrix}.$$

Note that $\{\delta\}_j$ ($j = 1, 2, \ldots, 2N$) must satisfy the continuity conditions (equations (4a) and (4b)), and therefore, they are not independent variables.

3. THE FIRST APPLICATION OF THE U-TRANSFORMATION

As the above equivalent system possesses cyclic periodicity due to repetition of the substructure units, the U-transformation can be used to decompose the equations of motion into a set of uncoupled equations. The theory of U-transformation was described in detail by Chan *et al.* [6], and therefore, only the pertinent points will be highlighted here.

In the present study, U-transformation is carried out for the displacements $\{\delta\}_j$, that is

$$\{q\}_m = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} \mathrm{e}^{-\mathrm{i}(j-1)m\psi}\{\delta\}_j, \quad m = 1, 2, \ldots, 2N, \tag{11a}$$

$$\{f\}_m = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} \mathrm{e}^{-\mathrm{i}(j-1)m\psi}\{F\}_j, \quad m = 1, 2, \ldots, 2N, \tag{11b}$$

where $\{q\}_m$ and $\{f\}_j$ are the transformed vectors, $\psi = \pi/N$ and $\mathrm{i} = \sqrt{-1}$. The system is anti-symmetric and we have

$$\{q\}_{2N-m} = \{\bar{q}\}_m, \quad m = 1, 2, \ldots, N \tag{12}$$

It can be shown readily that the inverse transformations can be defined as

$$\{\delta]_j = \frac{1}{\sqrt{2N}} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(j-1)m\psi}\{q\}_m, \quad j = 1, 2, \ldots, 2N, \tag{13a}$$

$$\{F\}_j = \frac{1}{\sqrt{2N}} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(j-1)m\psi}\{f\}_m, \quad j = 1, 2, \ldots, 2N. \tag{13b}$$

Substituting equations (13a) and (13b) into equations (2) and (9) for the total potential energy V and kinetic energy T, summing up $2N$ terms and noting the identity characteristic of U-transformation

$$\frac{1}{2N} \sum_{j=1}^{2N} \mathrm{e}^{\mathrm{i}(j-1)(m-n)\psi} = \begin{cases} 1, & m-n = 0, \ \pm 2N, \ \pm 4N, \ldots, \\ 0, & m-n = \text{other integers} \end{cases} \tag{14}$$

we have

$$V = \sum_{m=1}^{2N} \frac{1}{2}\{\bar{q}\}_m^{\mathrm{T}}[K]_{sub}\{q\}_m - \sum_{m=1}^{2N} \frac{1}{2}(\{\bar{q}\}_m^{\mathrm{T}}\{f\}_m + \{q\}_m^{\mathrm{T}}\{\bar{f}\}_m), \tag{15}$$

$$T = \sum_{m=1}^{2N} \frac{1}{2}\{\dot{\bar{q}}\}_m^{\mathrm{T}}[M]_{sub}\{\dot{q}\}_m. \tag{16}$$

As in the case of $\{\delta]_j$, the transformed vector $\{q\}_m$ can be written as

$$\{q\}_m \equiv \begin{Bmatrix} q_L \\ q_R \end{Bmatrix}_m. \tag{17}$$

Taking into account the continuity condition between adjacent substructure units, one can show that

$$\{q_R\}_m = e^{im\psi}\{q_L\}_m, \quad m = 1, 2, \ldots, 2N. \tag{18}$$

The above equation indicates that the corresponding components of $\{q_R\}_m$ and $\{q_L\}_m$ have the same amplitude but there is a phase difference of $m\psi$.

In matrix form, equation (18) can be rewritten as

$$\{q\}_m = [t]_m\{q_L\}_m, \quad m = 1, 2, \ldots, 2N, \tag{19}$$

where

$$[t]_m = \begin{bmatrix} I \\ e^{im\psi} I \end{bmatrix}$$

in which $[I]$ is the unit matrix of fourth order. It follows that $[t]_{2N-m} = [\bar{t}\,]_m$.

Using equation (19) to eliminate $\{q_R\}_m$ from (15) and (16), one can show that

$$V = \sum_{m=1}^{2N} \frac{1}{2}\{\bar{q}_L\}_m^{\mathrm{T}}[K]_m^*\{q_L\}_m - \sum_{m=1}^{2N} \frac{1}{2}(\{\bar{q}_L\}_m^{\mathrm{T}}\{f\}_m^* + \{q_L\}_m^{\mathrm{T}}\{\bar{f}\}_m^*), \tag{20}$$

$$T = \sum_{m=1}^{2N} \frac{1}{2}\{\dot{\bar{q}}_L\}_m^{\mathrm{T}}[M]_m^*\{\dot{q}_L\}_m, \tag{21}$$

where

$$[K]_m^* = [\bar{t}\,]_m^{\mathrm{T}}[K]_{sub}[t]_m,$$

$$[M]_m^* = [\bar{t}\,]_m^{\mathrm{T}}[M]_{sub}[t]_m,$$

$$\{f\}_m^* = [\bar{t}\,]_m^{\mathrm{T}}\{f\}_m = \begin{Bmatrix} 0 \\ f_{(1,m)} + f_{(1,m)}^0 \\ 0 \\ f_{(2,m)} \end{Bmatrix} e^{i\omega t}.$$

The explicit forms for $[K]_m^*$, $[M]_m^*$, $f_{(1,m)}$, $f_{(1,m)}^0$ and $f_{(2,m)}$ are given in Appendix A.

For a dynamic system, the Lagrange equations of motion [14] can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, \tag{22}$$

where q_i and Q_i denote the generalized co-ordinate and non-potential force respectively. In the present case, the loading can be regarded as potential forces. Therefore, the corresponding potential energy is included in V, and Q_i vanishes. Furthermore, q_i are elements of the transformed vector $\{q_L\}_m$.

Substituting the total potential energy V and kinetic energy T from equations (20) and (21) into equation (22), the equations of motion can be written as

$$[K]_m^*\{q_L\}_m + [M]_m^*\{\ddot{q}_L\}_m = \{f\}_m^*, \quad m = 1, 2, \ldots, N, 2N. \tag{23}$$

Expressing $\{q_L\}_m = [q_{(1,m)}, q_{(2,m)}, q_{(3,m)}, q_{(4,m)}]^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega t}$, one has

$$([K]_m^* - \omega^2 [M]_m^*) \begin{Bmatrix} q_{(1,m)} \\ q_{(2,m)} \\ q_{(3,m)} \\ q_{(4,m)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_{(1,m)} + f^0_{(1,m)} \\ 0 \\ f_{(2,m)} \end{Bmatrix}, \quad m = 1, 2, \ldots, N, 2N. \tag{24}$$

Note that the above equation is a set of independent linear simultaneous equations having four variables.

As the inertial terms only depend on transverse deformations ($q_{(2,m)}$ and $q_{(4,m)}$), we can eliminate $q_{(1,m)}$ and $q_{(3,m)}$, and the above equation becomes

$$\begin{bmatrix} k_{11,m} - \omega^2 M_1 & k_{12,m} \\ k_{21,m} & k_{22,m} - \omega^2 M_2 \end{bmatrix} \begin{Bmatrix} q_{(2,m)} \\ q_{(4,m)} \end{Bmatrix} = \begin{Bmatrix} f_{(1,m)} + f^0_{(1,m)} \\ f_{(2,m)} \end{Bmatrix}, \quad m = 1, 2, \ldots, N, 2N, \tag{25}$$

where

$$k_{=11,m} = k_{22,m} = \frac{2K_1 \sin^2 \alpha [(1 - \cos m\psi)(2\beta - \cos^2 \alpha) + 4\cos^2 \alpha]}{2\beta^2 (1 - \cos m\psi) + 4\beta \cos^2 \alpha + \cos^4 \alpha},$$

$$k_{=12,m} = \bar{k}_{21,m} = -\frac{2K_1 \sin^2 \alpha (1 + \mathrm{e}^{-\mathrm{i}m\psi})[2\cos^2 \alpha + \beta(1 - \cos m\psi)]}{2\beta^2 (1 - \cos m\psi) + 4\beta \cos^2 \alpha + \cos^4 \alpha}$$

with $\beta = K_1/K_2$.

The above equation can be easily solved and the solution can be expressed as

$$\begin{Bmatrix} q_{(2,m)} \\ q_{(4,m)} \end{Bmatrix} = \frac{1}{\Delta_m} \begin{bmatrix} k_{22,m} - \omega^2 M_2 & -k_{12,m} \\ -k_{21,m} & k_{11,m} - \omega^2 M_1 \end{bmatrix} \begin{Bmatrix} f_{(1,m)} + f^0_{(1,m)} \\ f_{(2,m)} \end{Bmatrix} \quad (\text{if } \Delta_m \neq 0)$$

$$m = 1, 2, \ldots, 2N, \tag{26a}$$

where

$$\Delta_m = (k_{11,m} - \omega^2 M_1)(k_{22,m} - \omega^2 M_2) - k_{12,m} k_{21,m}. \tag{26b}$$

One can then obtain the transverse deformations by carrying out the inverse U-transformation:

$$v_{(1,j)} = \frac{1}{\sqrt{2N}} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(j-1)m\psi} \frac{1}{\Delta_m} [(k_{22,m} - \omega^2 M_2)(f_{(1,m)} + f^0_{(1,m)}) - k_{12,m} f_{(2,m)}], \tag{27a}$$

$$v_{(2,j)} = \frac{1}{\sqrt{2N}} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(j-1)m\psi} \frac{1}{\Delta_m} [-k_{21,m} - (f_{(1,m)} + f^0_{(1,m)}) + (k_{11,m} - \omega^2 M_1) f_{(2,m)}], \tag{27b}$$

where $v_{(1,j)}$ and $v_{(2,j)}$ denote the amplitudes of the transverse displacements for nodes $(1, j)$ and $(2, j)$ respectively. In the above equations, $f^0_{(1,m)}$ is a function of P_k (see Appendix A) and it can be determined by considering the constraint condition at the support. Mathematically, the constraints can be written as

$$v_{(1,(s-1)p+1)} = 0, \quad s = 1, 2, \ldots, 2n. \tag{28}$$

From equation (27a), we have

$$\sum_{k=1}^{2n} \beta_{s,k} P_k + V_s = 0, \quad s = 1, 2, \ldots, 2n \tag{29}$$

in which

$$\beta_{s,k} \equiv \frac{1}{2N} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(s-k)mp\psi} \frac{1}{\Delta_m} (k_{22,m} - \omega^2 M_2),$$

$$V_s = \frac{1}{\sqrt{2N}} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(s-k)mp\psi} \frac{1}{\Delta_m} [(k_{22,m} - \omega^2 M_2) f_{(1,m)} - k_{12,m} f_{(2,m)}].$$

Here, V_s represents the transverse displacement of the sth supported node caused by the external harmonic excitation for the equivalent system without supports. It can be verified that V_s $(s = 1, 2, \ldots, 2n)$ possess antisymmetry, i.e.,

$$V_{2n+2-s} = -V_s, \quad s = 2, 3, \ldots, n, \tag{30a}$$

$$V_1 = V_{n+1} = 0. \tag{30b}$$

Equation (29) is a set of linear simultaneous equations with unknowns P_k $(k = 1, 2, \ldots, 2n)$. The coefficients $\beta_{s,k}$ $(s, k = 1, 2, \ldots, 2n)$ possess cyclic periodicity as well, namely

$$\beta_{1,1} = \beta_{2,2} = \cdots = \beta_{2n,2n}, \tag{31a}$$

$$\beta_{s,1} = \beta_{s+1,2} = \cdots = \beta_{2n,2n-s+1} = \beta_{1,2n-s+2} = \cdots = \beta_{s-1,2n}, \quad s = 2, 3, \ldots, 2n. \tag{31b}$$

4. THE SECOND APPLICATION OF THE U-TRANSFORMATION

As equation (29) possesses cyclic periodicity due to the repetition of the spans, U-transformation can be performed to uncouple the equation to form a set of single-degree-of-freedom equations. These equations can then provide us with the explicit solutions of the generalized support reactions.

Let the U-transformation be

$$Q_r = \frac{1}{\sqrt{2n}} \sum_{s=1}^{2n} \mathrm{e}^{-\mathrm{i}(s-1)r\varphi} P_s, \quad r = 1, 2, \ldots, 2n. \tag{32a}$$

The inverse U-transformation is then

$$P_s = \frac{1}{\sqrt{2N}} \sum_{r=1}^{2n} \mathrm{e}^{\mathrm{i}(s-1)r\varphi} Q_r, \quad s = 1, 2, \ldots, 2n \tag{32b}$$

with $\varphi = \pi/n = p\psi$.

Premultiplying (29) by $(1/\sqrt{2n}) \sum_{s=1}^{2n} \mathrm{e}^{-\mathrm{i}(s-1)r\varphi}$, we have

$$\sum_{k=1}^{2n} \beta_{k,1} \mathrm{e}^{-\mathrm{i}(k-1)r\varphi} Q_r + b_r = 0, \quad r = 1, 2, \ldots, 2n, \tag{33}$$

where

$$\beta_{k,1} = \frac{1}{2N} \sum_{m=1}^{2N} \mathrm{e}^{\mathrm{i}(k-1)m\varphi} \frac{1}{\Delta_m} (k_{22,m} - \omega^2 M_2), \tag{34a}$$

$$b_r = \frac{1}{\sqrt{2n}} \sum_{s=1}^{2n} \mathrm{e}^{-\mathrm{i}(s-1)r\varphi} V_s. \tag{34b}$$

Obviously, the above equation is a set of single-degree-of-freedom equations. Taking into account the antisymmetric property, one can show that

$$b_r = -\frac{2i}{\sqrt{2n}} \sum_{s=2}^{n} \sin\left[(s-1)r\varphi\right] V_s \tag{34c}$$

and

$$b_{2n-r} = \bar{b}_r, \quad b_n = b_{2n} = 0. \tag{34d}$$

Therefore, equation (33) can be rewritten as

$$a_r(\omega)Q_r + b_r = 0, \quad r = 1, 2, \ldots, n-1, \tag{35a}$$

$$Q_{2n-r} = \bar{Q}_r, \quad r = 1, 2, \ldots, n-1, \tag{35b}$$

$$Q_n = Q_{2n} = 0, \tag{35c}$$

where

$$a_r(\omega) = \frac{1}{p} \sum_{k=1}^{p} \frac{k_{22, r+(k-1)2n} - \omega^2 M_2}{\Delta_{r+(k-1)2n}}.$$

If $a_r(\omega) \neq 0$ ($r = 1, 2, \ldots, n-1$), the solution for Q_r in equation (35a) can be expressed as

$$Q_r = -\frac{b_r}{a_r(\omega)}. \tag{36}$$

When the specific structure and loading parameters are given, Q_r can be calculated and the support reactions as well as the nodal displacements can be found.

Recalling the definitions of both generalized support reactions $f^0_{(1,m)}$ (Appendix A) and Q_r, one can establish a direct relation between $f^0_{(1,m)}$ and Q_r

$$f^0_{(1, m+(k-1)2n)} = \frac{1}{\sqrt{p}} Q_m, \quad m = 1, 2, \ldots, 2n, \quad k = 1, 2, \ldots, p. \tag{37}$$

In using the above equation, one can calculate the nodal deformations without first finding the support reactions. It will save considerable computing effort.

If $a_r(\omega) = 0$, Q_r will approach infinity and resonance will occur, that is

$$\sum_{k=1}^{p} \frac{k_{22, r+(k-1)2n} - \omega^2 M_2}{\Delta_{r+(k-1)2n}} = 0, \quad r = 1, 2, \ldots, n-1. \tag{38}$$

It is obvious that the roots, ω, of equation (38) are the natural frequencies of the system. Note that these vibration modes have non-zero support reactions.

For certain modes of vibration, the support reactions are identically zero. Therefore, it is not possible to calculate the natural frequencies using the above approach. The corresponding frequency equation for these cases can be obtained by setting equation (26b) to zero:

$$M_1 M_2 \omega^4 - (M_1 k_{22,m} + M_2 k_{11,m})\omega^2 + k_{11,m}k_{22,m} - k_{12,m}k_{21,m} = 0, \quad m = n, 2n, \ldots, pn, \tag{39}$$

where m is the number of half waves of the natural mode of the original truss. The natural frequencies can be obtained by solving equation (39).

5. EXAMPLES

5.1. NATURAL VIBRATION

Consider a continuous Warren truss having $2n$ substructure units and $n+1$ supports. The structural parameters are

$$K_1 = K_2 = K, \qquad M_1 = M_2 = M \qquad \alpha = \frac{\pi}{3}, \qquad p = 2 \tag{40a}$$

and

$$N = 2n, \qquad \psi = \frac{\pi}{2n}, \quad \varphi = \frac{\pi}{n}, \quad \beta = 1, \tag{40b}$$

where n is an arbitrary positive integer.

Substituting these parameters into equation (38) yields

$$(K_1(r) - \Omega)[(K_1(r+2n) - \Omega)^2 - K_2(r+2n)] + (K_1(r+2n) - \Omega)$$
$$[(K_1(r) - \Omega)^2 - K_2(r)] = 0, \quad r = 1, 2, \ldots, n-1, \quad n = 2, 3, \ldots, \infty, \tag{41}$$

where

$$K_1(m) = \frac{6(11 - 7\cos m\psi)}{49 - 32\cos m\psi},$$

$$K_2(m) = \frac{288(1 + \cos m\psi)(3 - 2\cos m\psi)^2}{(49 - 32\cos m\psi)^2},$$

$$\Omega \equiv \frac{M\omega^2}{K}$$

and Ω denotes the non-dimensional frequency parameter. From the above equation, one can obtain the natural frequencies and the results for various numbers of spans are tabulated in Table 1. There are $3(n-1)$ natural frequencies altogether.

One can further find the frequencies corresponding to the cases when the support reactions are identically zero. From equation (39), one can obtain

$$(K_1(m) - \Omega)^2 - K_2(m) = 0, \quad m = n,\ 2n. \tag{42}$$

The solutions to the above equations are

$$\Omega = \tfrac{4}{3} \quad (\text{for } m = 2n), \qquad \tfrac{6}{49}(11 \mp 6\sqrt{2}) \quad (\text{for } m = n). \tag{43}$$

The corresponding frequencies are

$$\omega = 0{\cdot}55490966\sqrt{\frac{K}{M}}, \qquad 1{\cdot}1547005\sqrt{\frac{K}{M}}, \qquad 1{\cdot}5446530\sqrt{\frac{K}{M}} \tag{44}$$

These natural frequencies are independent of n and in agreement with those for the portion of truss between two adjacent supports, i.e., the case $n = 1$. The numerical results are also shown in Table 1. They represent the lower limits of the three frequency bands.

The total number of the natural frequencies is equal to $3n$, which is in agreement with the number of degrees of freedom for the truss considered. The results in Table 1 shown that all natural frequencies lie in three frequency bands. If n approaches infinity, each band is full of natural frequencies.

TABLE 1

Natural frequency ω of the trusses with $p = 2$

n	$k^{\dagger}$ 1		2		3	
2	0·55490966	0·5822044	1·1547005	1·158549	1·5446530	1·545193
3	0·55490966 0·5670646	0·6016117	1·1547005 1·156978	1·159699	1·5446530 1·544889	1·545597
4	0·55490966 0·5617220	0·5822044 0·6113639	1·1547005 1·156144	1·158549 1·160089	1·5446530 1·544785	1·545193 1·545807
5	0·55490966 0·5592594 0·5724387	0·5935839 0·6166260	1·1547005 1·155682 1·157641	1·159298 1·160267	1·5446530 1·544737 1·544996	1·545428 1·545922
6	0·55490966 0·5579257 0·5670646	0·5822044 0·6016117 0·6197189	1·1547005 1·155406 1·156978	1·158549 1·159699 1·160363	1·5446530 1·544711 1·544889	1·545193 1·545597 1·545990
30	0·55490966 0·5550296 0·5553907 0·5559930 0·5568375 0·5579257 0·5592594 0·5608393 0·5626669 0·5647421 0·5670646 0·5696318 0·5724387 0·5754783 0·5787387	0·5822044 0·5858539 0·5896588 0·5935839 0·5975856 0·6016117 0·6054014 0·6094850 0·6131870 0·6166260 0·6197189 0·6223847 0·6245478 0·6261437 0·6271220	1·1547005 1·154731 1·154821 1·154968 1·155165 1·155406 1·155682 1·155985 1·156308 1·156641 1·156978 1·157313 1·157641 1·157959 1·158262	1·158549 1·158818 1·159068 1·159298 1·159508 1·159699 1·159869 1·160021 1·160153 1·160267 1·160363 1·160440 1·160500 1·160543 1·160568	1·5446530 1·544655 1·544662 1·544673 1·544690 1·544711 1·544737 1·544768 1·544803 1·544844 1·544889 1·544940 1·544996 1·545057 1·545123	1·545193 1·545268 1·545346 1·545428 1·545512 1·545597 1·545682 1·545766 1·545847 1·545922 1·545990 1·546049 1·546097 1·546133 1·546155
∞	[0·55490966, 0·627451292)[‡]		[1·1547005, 1·1605769)		[1·5446530, 1·5461623)	

Note: Multiplier — $\sqrt{K/M}$.
[†] k denotes the ordinal number of the frequency band.
[‡] $[\omega_L, \omega_U)$ represents $\omega_L \leqslant \omega < \omega_U$.

The upper and lower limits of the frequency bands can be obtained by solving equation (41) with $r = 0$, n. The natural frequencies can be shown to be

$$\omega_{1L} = 0{\cdot}55490966\sqrt{\frac{K}{M}}, \qquad \omega_{1U} = 0{\cdot}62745192\sqrt{\frac{K}{M}}, \tag{45a}$$

$$\omega_{2L} = 1{\cdot}1547005\sqrt{\frac{K}{M}}, \qquad \omega_{2U} = 1{\cdot}1605769\sqrt{\frac{K}{M}}, \tag{45b}$$

$$\omega_{3L} = 1{\cdot}5446530\sqrt{\frac{K}{M}}, \qquad \omega_{3U} = 1{\cdot}5461623\sqrt{\frac{K}{M}}, \tag{45c}$$

where ω_{kL} and ω_{kU} ($k = 1, 2, 3$) denote the lower and upper limits of the kth frequency band respectively. Note that the lower limits agree with those obtained using equation (44).

In Table 1, the numerical results show that the natural frequencies of the continuous truss with different assigned numbers of spans do not include the upper limit of the three frequency bands, but they can approach every upper bound of the bands as a limit when n approaches infinity.

When n is a large number, the natural frequencies are densely distributed in each frequency band. It is not easy to find the dense natural frequencies by conventional eigenvalue numerical methods. In the present approach, the frequency equation of the continuous truss with n spans is uncoupled to form n frequency equations. The natural frequencies obtained from each one are dispersed, namely, they lie in different frequency bands. Consequently, accurate natural frequencies can be easily found no matter how large n is.

One can also easily calculate the vibration mode shape. If Ω_r denotes a root for Ω of the rth equation in equation (42) for a given r, the non-trivial solution for Q_m $(m = 1, 2, \ldots, 2n)$ can be expressed as

$$Q_r = \mathrm{i}A, \quad Q_{2n-r} = -\mathrm{i}A, \tag{46}$$

with the other Q_m vanishing where A indicates an arbitrary real constant with the unit of force.

From equation (37) we can obtain

$$f^0_{(1,r)} = f^0_{(1,r+2n)} = \frac{\mathrm{i}A}{\sqrt{2}}, \tag{47a}$$

$$f^0_{(1,2n-r)} = f^0_{(1,4n-r)} = -\frac{\mathrm{i}A}{\sqrt{2}} \tag{47b}$$

with the other $f^0_{(1,m)}$ vanishing.

Substituting equations (47a, b), (40a, b) and $f_{(1,m)} = f_{(2,m)} = 0$ into equation (27a, b), one can show that

$$v_{(1,j)} = -\frac{A}{\sqrt{2n}K} \sum_{m=r,\,r+2n} \sin[(j-1)m\psi] \frac{K_1(m) - \Omega_r}{[K_1(m) - \Omega_r]^2 - K_2(m)}, \tag{48a}$$

$$v_{(2,j)} = -\frac{A}{\sqrt{2n}K} \sum_{m=r,\,r+2n} \sin\left[\left(j - \frac{1}{2}\right)m\psi\right] \cos\left(\frac{1}{2}m\psi\right) \frac{K_3(m)}{[K_1(m) - \Omega_r]^2 - K_2(m)}, \tag{48b}$$

where $K_1(m)$ and $K_2(m)$ have been defined in equation (41), and

$$K_3(m) = \frac{24(3 - 2\cos m\psi)}{49 - 32\cos m\psi} \tag{49}$$

with $\psi = \pi/2n$.

It can be proved that when r is odd, the mode shown in equation (48a, b) is symmetric, i.e., $v_{(1,2n+2-j)} = v_{(1,j)}$ and $v_{(2,2n+1-j)} = v_{(2,j)}$; when r is even, the mode is antisymmetric, i.e., $v_{(1,2n+2-j)} = -v_{(1,j)}$ and $v_{(2,2n+1-j)} = -v_{(2,j)}$.

5.2. FORCED VIBRATION

Consider now a Warren truss having six substructures and four roller supports subjected to a harmonic loading $F\mathrm{e}^{\mathrm{i}\omega t}$ acting at the centre node (Figure 4).

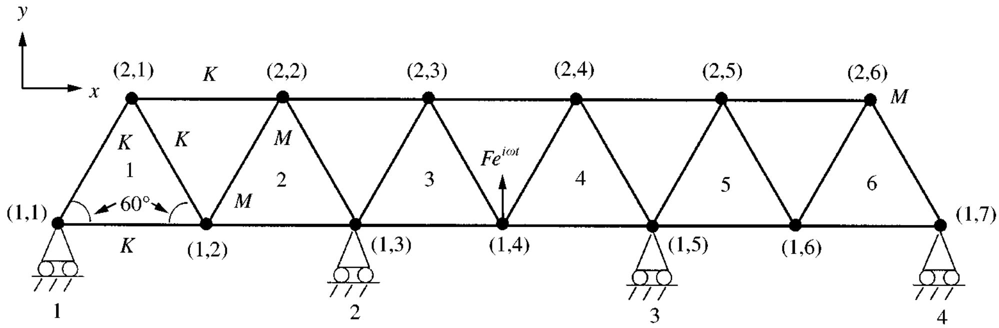

Figure. 4. Plane truss with six substructures and four supports subjected to a harmonic force at the centre node.

The structural parameters are given as

$$N = 6, \quad n = 3, K_1 = K_2 = K, \quad M_1 = M_2 = M, \quad \alpha = \frac{\pi}{3} \tag{50a}$$

and

$$\psi = \frac{\pi}{6}, \quad \varphi = \frac{\pi}{3}, \quad \beta = 1, \quad p = 2. \tag{50b}$$

The amplitudes of the nodal loads can be expressed as

$$F_{(1,4)} = F, \quad F_{(1,j)} = 0, \quad j \neq 4, \tag{51a}$$

$$F_{(2,j)} = 0, \quad j = 1, 2, \ldots, 6. \tag{51b}$$

One can show readily that

$$f_{(1,m)} = -\frac{\mathrm{i}}{\sqrt{3}} \sin \frac{m\pi}{2} F, \quad m = 1, 2, \ldots, 12, \tag{52a}$$

$$f_{(2,m)} = 0, \quad m = 1, 2, \ldots, 12. \tag{52b}$$

Substituting the relevant formulae in equation (27a) with $j = 4$, the amplitude response function for the loaded node can be found in an explicit form as

$$v_{(1,4)} = H(\Omega) \frac{F}{K}, \tag{53}$$

where

$$H(\Omega) = \frac{4}{3} \frac{(K_1(1) - \Omega)(K_1(7) - \Omega)}{(K_1(1) - \Omega)[(K_1(7) - \Omega)^2 - K_2(7)] + (K_1(7) - \Omega)[(K_1(1) - \Omega)^2 - K_2(1)]} + \frac{K_1(3) - \Omega}{3[(K_1(3) - \Omega)^2 - K_2(3)]}. \tag{54}$$

The frequency response curve, $v_{(1,4)}$ versus Ω, is plotted in Figure 5, showing five resonance frequencies. When Ω is close to these points, $H(\Omega)$ will approach infinity. These

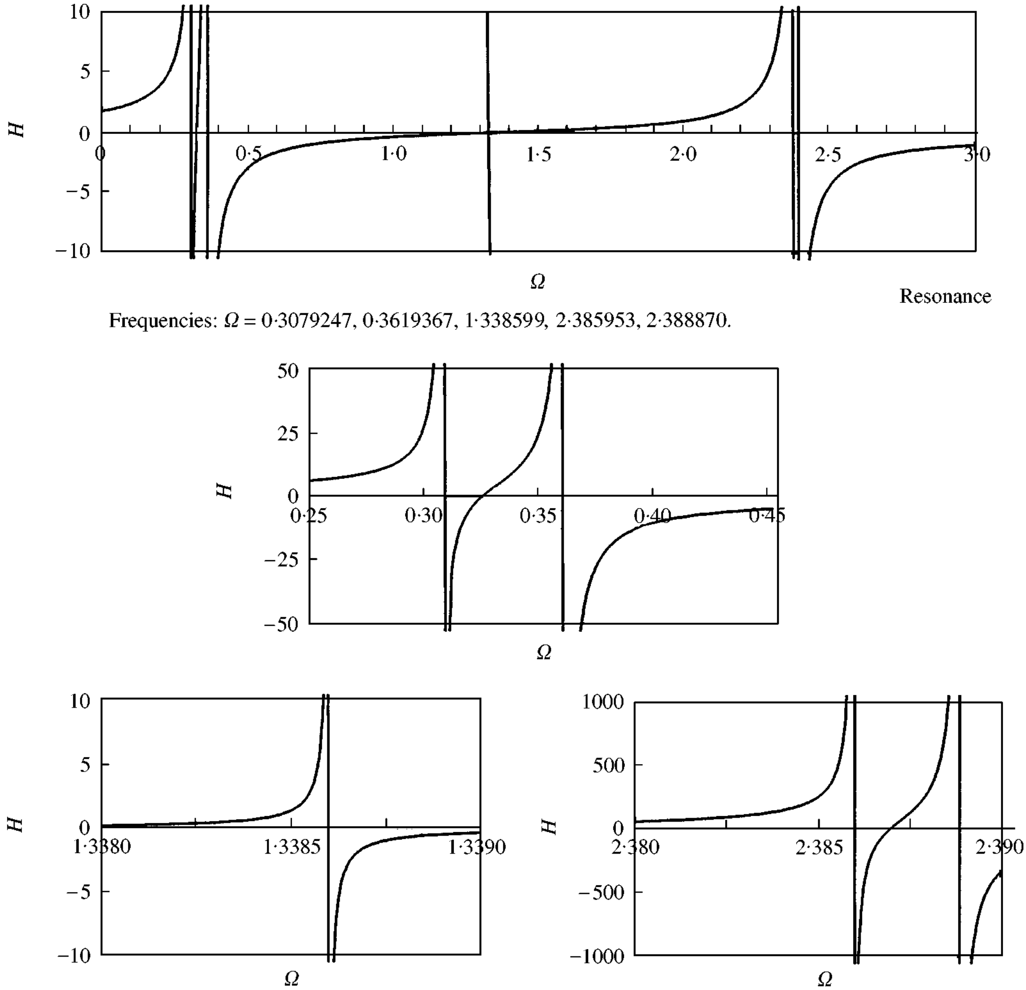

Figure. 5. Frequency response curve, $H = v_{(1,4)}\, K/F$, $\Omega = \omega^2\, M/K$.

resonance frequencies can also be determined by setting the denominators of equation (53) to zero, that is

$$(K_1(1) - \Omega)[(K_1(7) - \Omega)^2 - K_2(7)] + (K_1(7) - \Omega)[(K_1(1) - \Omega)^2 - K_2(1)] = 0, \quad (54a)$$

$$(K_1(3) - \Omega)^2 - K_2(3) = 0. \quad (54b)$$

The roots for Ω in equation (54b) are $\frac{6}{49}(11 \mp 6\sqrt{2})$ and the roots in equation (54a) are 0·3619367, 1·338599 and 2·388870. These five resonance frequencies correspond to the symmetric modes of the truss. It is pointed out that there are four other resonance frequencies (0·5670646 $\sqrt{K/M}$, 1·1547005 $\sqrt{K/M}$, 1·1595699 $\sqrt{K/M}$ and 1·544889 $\sqrt{K/M}$) at which antisymmetric modes exist. For these mode shapes, the transverse deformation is zero at the loaded node, and therefore the generalized force exciting the mode is zero. No resonant response can be generated and so resonance peaks at these frequencies do not appear on the diagram.

One can check the present solutions by comparing its results with those of the static case. In the static case, Ω will approach zero, $v_{(1,4)}$ will approach $H(0)\ F/K$, where

$$H(0) = \frac{4}{3}\frac{K_1(1)K_1(7)}{K_1(1)[K_1^2(7) - K_2(7)] + K_1(7)[K_1^2(1) - K_2(1)]} + \frac{K_1(3)}{3[K_1^2(3) - K_2(3)]} = \frac{353}{210}. \tag{55}$$

This result is in agreement with the exact static displacement obtained by Cai *et al.* [12]. It is well known that when Ω approaches infinity, $v_{(1,4)}$ approaches zero.

6. CONCLUSIONS

In the present study, the application of U-transformation has been extended from static analysis to dynamic analysis of continuous Warren trusses. The trusses considered belong to the category of bi-periodic structures. In order to fully utilize the property of bi-periodicity, the proposed analysis method requires the application of U-transformation twice. The governing equation of harmonic vibration is derived and uncoupled to form a set of single-degree-of-freedom equations, which lead to a set of frequency equations and frequency response function. There are two kinds of frequency equations corresponding to support reactions, vanishing and non-vanishing respectively. It is well known that the natural frequencies of a periodic structure lie densely in the frequency bands. It is not easy to accurately calculate these dense natural frequencies by using numerical methods. In this study, however, the natural frequencies corresponding to each frequency equation for an assigned r or m lie in different frequency bands (i.e., they are dispersed), so accurate frequencies can be easily found by the present method.

The proposed method is applicable to the dynamic analysis of other types of periodic trusses with periodic supports.

ACKNOWLEDGMENT

The constructive suggestions made by the referees are acknowledged.

REFERENCES

1. Y. K. LIN and T. J. MCDANIEL 1969 *Journal of Engineering for Industry* **91**, 1133–1141. Dynamics of beam-type periodic structures.
2. G. S. GUPTA 1972 *Journal of Sound and Vibration* **20**, 39–49. Propagation of flexural waves in doubly periodic structures.
3. D. J. MEAD 1975 *Journal of Sound and Vibration* **40**, 1–18. Wave propagation and natural modes in periodic system, I. Monocoupled systems.
4. D. J. MEAD 1975 *Journal of Sound and Vibration* **40**, 19–39. Wave propagation and natural modes in periodic system, II. Multicoupled systems with and without damping.
5. T. J. MCDANIEL and M. J. CARROLL 1982 *Journal of Sound and Vibration* **81**, 311–335. Dynamics of bi-periodic structures.
6. C. W. CAI, H. C. CHAN and Y. K CHEUNG 1998 *Journal of Engineering Mechanics, American Society of Civil Engineers* **124**, 836–841. Exact method for static and natural vibration analyses of bi-periodic structures.
7. H. C. CHAN, Y. K. CHEUNG and C. W. CAI 1998 *Journal of Sound and Vibration* **218**, 29–44. Exact solution for vibration analysis of rectangular cable networks with periodically distributed supports.

8. C. W. CAI, Y. K. CEUNG and H. C. CHAN 1989 *Journal of Sound and Vibration* **133**, 139–150. Transverse vibration analysis of plane trusses by analytical method.
9. C. W. CAI, Y. K. CHEUNG and H. C. CHAN 1988 *Journal of Sound and Vibration* **123**, 461–472. Dynamic response of infinite continuous beams subjected to a moving force—an exact method.
10. C. W. CAI, Y. K. CHEUNG and H. C. CHAN 1995 *Journal of Applied Mechanics, American Society of Mechanical Engineers* **62**, 141–149. Mode localization phenomena in nearly periodic system.
11. Y. K. CHEUNG, H. C. CHAN and C. W. CAI 1989 *Journal of Engineering Mechanics, American Society of Civil Engineers* **115**, 415–434. Exact method for static analysis of periodic structures.
12. C. W. CAI, H. C. CHAN and J. K. LIU 2000 *Journal of Engineering Mechanics, American Society of Civil Engineers* **126,** 333–339. Analytical solution for plane trusses with equidistant supports.
13. H. C. CHAN, C. W. CAI and Y. K. CHEUNG 1998 *Exact Analysis of Structures with Periodicity Using U-transformation.* Singapore: World Scientific Publishing Co.
14. L. MEIROVITCH 1997 *Principles and Techniques of Vibrations.* Englewood Cliffs, NJ: Prentice-Hall, 88–91.

APPENDIX A

After transformation, the stiffness matrix, mass matrix and force vector are as follows.

(*a*) *Stiffness matrix*

$$[K]^*_m =$$

$$\begin{bmatrix} 2K_1(1-\cos m\psi)+2K_2\cos^2\alpha & 0 & -K_2\cos^2\alpha(1+e^{-im\psi}) & -K_2\sin\alpha\cos\alpha(1-e^{-im\psi}) \\ 0 & 2K_2\sin^2\alpha & -K_2\sin\alpha\cos\alpha(1-e^{-im\psi}) & -K_2\sin^2\alpha(1+e^{-im\psi}) \\ -K_2\cos^2\alpha(1+e^{im\psi}) & -K_2\sin\alpha\cos\alpha(1-e^{im\psi}) & 2K_1(1-\cos m\psi)+2K_2\cos^2\alpha & 0 \\ -K_2\sin\alpha\cos\alpha(1-e^{im\psi}) & -K_2\sin^2\alpha(1+e^{im\psi}) & 0 & 2K_2\sin^2\alpha \end{bmatrix} \tag{A1}$$

(*b*) *Mass matrix*

$$[M]^*_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 \end{bmatrix}. \tag{A2}$$

Note that $[K]^*_m$ and $[M]^*_m$ are Hermitian matrices, i.e., $[\bar{K}]^{*T}_m \equiv [K]^*_m$ and $[\bar{M}]^{*T}_m \equiv [M]^*_m$.

(*c*) *Force vector*

The force vector can be defined as

$$\{f\}_m = [0 \quad f_{(1,m)} + f^0_{(1,m)} \quad 0 \quad f_{(2,m)} \quad 0 \quad 0 \quad 0 \quad 0]^T e^{i\omega t}, \tag{A3}$$

where

$$f_{(1,m)} = \frac{-2i}{\sqrt{2N}} \sum_{k=2}^{N} \sin(k-1)m\psi F_{(1,k)},$$

$$f_{(2,m)} = \frac{-2i}{\sqrt{2N}} e^{im\psi/2} \sum_{k=1}^{N} \sin\left(k-\frac{1}{2}\right)m\psi F_{(2,k)},$$

$$f^0_{(1,m)} = \frac{1}{\sqrt{2N}} \sum_{k=1}^{2n} e^{-i(k-1)pm\psi} P_k.$$

In the condensed form, we have

$$\{f\}_m^* = \begin{Bmatrix} 0 \\ f_{(1,m)} + f_{(1,m)}^0 \\ 0 \\ f_{(2,\,m)} \end{Bmatrix} e^{i\omega t}\,. \tag{A4}$$